Equations of families of Complex Algebraic Surfaces with $p_g = q = 2$

(Joint work with Fabrizio Catanese)

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UNIVERSITÀ DEGLI STUDI DI GENOVA
UNIVERSITÄT BAYREUTH

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Introduction

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What is a complex manifold?

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What is a complex manifold?

Roughly speaking, a generalization of a Riemann Surface.

Riemann Surface: Definition

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Two complex charts $\varphi_i : U_i \to V_i$, $i = 1, 2$ are said to be *holomorphically compatible* if the map

$$\varphi_2 \circ \varphi_1^{-1} : \varphi_1(U_1 \cap U_2) \to \varphi_2(U_1 \cap U_2)$$

is biholomorphic.

Riemann Surface: Definition

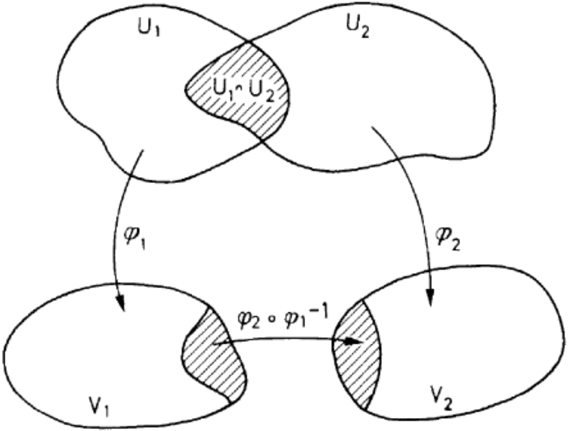

Figure: Upstairs open sets in X. Downstairs open sets in $\mathbb{C}$.

Riemann Surface: Formal Definition

A *complex atlas* on X is a collection

$$\mathcal{U} := \{\varphi_i : U_i \to V_i \mid i \in I\}$$

of charts which are holomorphically compatible and which cover X, i.e.,

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Definition

By a *complex structure* on X we mean an equivalence class $[\mathcal{U}]$ of analytically equivalent atlases on X.

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Example

- The complex plane $\mathbb{C}$: canonical atlas $\mathcal{U} = \{id : \mathbb{C} \to \mathbb{C}\}$
- The Riemann Sphere $\mathbb{P}^1_{\mathbb{C}} := \mathbb{C} \cup \infty$
- Complex Tori

$$T = \mathbb{C}/\Lambda, \qquad \Lambda \text{ lattice in } \mathbb{R}^2 = \mathbb{C}$$

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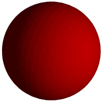

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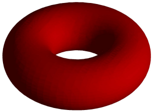

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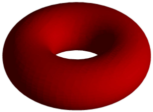

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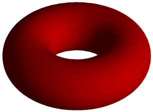

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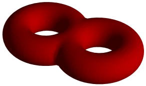

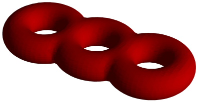

Complex Tori: Construction

Suppose $\omega_1, \omega_2 \in \mathbb{C}$ are linearly independent over $\mathbb{R}$.

$$\Lambda := \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 = \{n\omega_1 + m\omega_2 : \ n, m \in \mathbb{Z}\}$$

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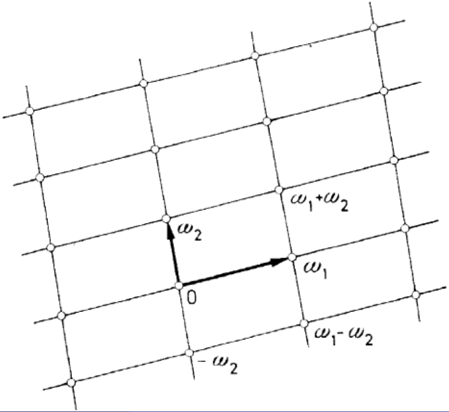

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By using the canonical projection

$$p \colon \mathbb{C} \to T$$

we can endow T with a complex structure in a natural way.

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Question

Is there a way to parametrize the family of all complex tori (up to isomorphism)?

Fact

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Suppose $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$ *and* $Im(\tau) > O$. *Let*

$$\tau' := \frac{a\tau + b}{c\tau + d}$$

Then, the tori $T(\tau)$ *and* $T(\tau')$ *are isomorphic. Moreover, the converse holds.*

The Moduli Space of Complex Tori

The space parametrizing the isomorphism classes of complex tori is then

$$\mathfrak{H} := \mathbb{H}/SL(2,\mathbb{Z}),$$

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Remark

Classically, this is the very first example of a *moduli space*.

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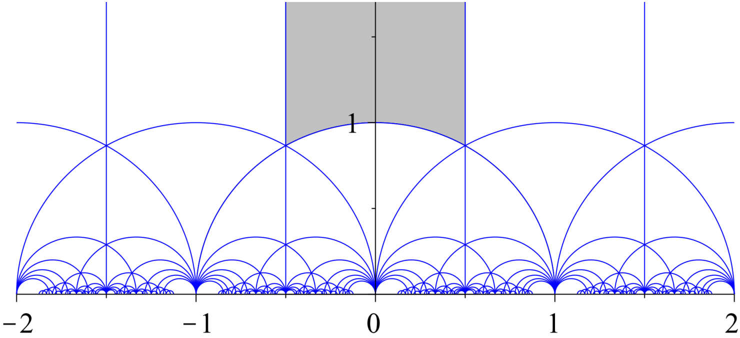

Figure: Fundamental domain of $\mathfrak{H}$, given by the grey part together with the boundary on the left plus half the arc on the bottom (including the point in the middle).

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Generalizing the concept of holomorphicity to functions

$$f = f(z_1, \dots, z_n) \colon U \subset \mathbb{C}^n \to \mathbb{C},$$

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Definition (Complex Manifold)

A complex manifold is a pair $(X, [\mathcal{U}])$, where X is as above and $[\mathcal{U}]$ is a given complex structure on X.
We say that n is the *(complex) dimension* of X.

Complex Manifolds: some remarks

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Riemann Surfaces are complex manifolds of dimension 1.

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Even if one knows that X is projective, it is not easy in general to find such global equations.

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A surface S is said to be *minimal* if it does not contains any (-1) curve, i.e. submanifold C of dimension 1 such that $C \cong \mathbb{P}^1, \quad C^2 = -1.$

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- the self-intersection of the canonical divisor K^2_S

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A surface S is said to be of *general type* if $\kappa(S) = 2$.

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$p_g = q = 2$: many contributions by several authors, but still widely open

Description of results from a joint work with Fabrizio Catanese

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- $\lambda \in \mathbb{C}$.

$p_g = q = 2$: CHPP surfaces

Theorem (-, Catanese)

The CHPP surfaces form an irreducible connected component of the moduli space of surfaces of general type with $K_S^2 = 5$, $p_g(S) = q(S) = 2$, and Albanese map $\alpha : S \to A = Alb(S)$ of degree $d = 3$.
It coincides with the component found by Penegini-Polizzi in 2013 and contains the so-called Chen-Hacon surfaces.

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PP4 surfaces: $p_g(S) = q(S) = 2,\ K_S^2 = 6,\ d = 4$
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- $\mu \in \mathbb{C}$.

Theorem (-,Catanese)

The family of PP4 surfaces provides an irreducible, connected component of the moduli space of surfaces of general type with $p_g = q = 2$, $K^2 = 6$ and Albanese map of degree 4.
It coincides with the component found by Penegini-Polizzi in 2014.

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Theorem (Catanese, Sernesi)

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This is the kernel of the isogeny

$$\Phi_D : A' \to A'^{\vee} := A$$

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- $\mathcal{H}_D/\mathcal{C} \cong G$.

Therefore, $\mathbb{P}(V)$ is a projective representation of G.

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Our method consists in describing a surface

$$S' \subset \mathbb{P}^{\delta-1} \times A' = \mathbb{P}(V) \times A',$$

which is G-invariant for the G-action of product type on $\mathbb{P}(V) \times A'$.

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Finally, we obtain S as the free quotient $S := S'/G$.

Thanks for listening!